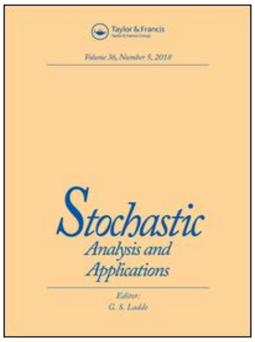

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Complete convergence theorems for weighted row sums from arrays of random elements in Rademacher type *p* and martingale type *p* Banach spaces

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ABSTRACT

In this correspondence, complete convergence theorems are established for weighted row sums $\{\sum_{i=1}^{j} a_{n,i}V_{n,i}, 1 \leq j \leq k_n, n \geq 1\}$ from arrays $\{V_{n,i}, 1 \leq i \leq k_n, n \geq 1, k_n \to \infty\}$ of random elements taking values in real separable Rademacher type p $(1 \leq p \leq 2)$ Banach spaces as well as real separable martingale type p $(1 \leq p \leq 2)$ Banach spaces. It is assumed that $\sup\{E\|V_{n,i}\|^p : 1 \leq i \leq k_n, n \geq 1\} < \infty$. A version of the Rademacher type p complete convergence theorem is also established with random variable weights. Illustrative examples are included.

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1. Introduction

The concept of complete convergence for a sequence of (real-valued) random variables was introduced by Hsu and Robbins [1] as follows. A sequence of random variables $\{U_n, n \geq 1\}$ is said to *converge completely* to 0 if

$$\sum_{n=1}^{\infty} P(|U_n| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0.$$

This implies by the Borel–Cantelli lemma that $U_n \to 0$ almost surely (a.s.). The converse is true if $\{U_n, n \geq 1\}$ is a sequence of independent random variables. A sequence of Banach space valued random elements is said to *converge completely* to the 0 element of the Banach space if the corresponding sequence of norms converges completely to 0.

Hsu and Robbins [1] and Erdös [2] investigated complete convergence for the sequence of arithmetic means of independent and identically distributed (i.i.d.) random

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variables. The Hsu–Robbins–Erdös result is formulated as follows where the sufficiency half is due to Hsu and Robbins [1] and the necessity half is due to Erdös [2].

Theorem 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables. Then $\frac{\sum_{i=1}^{n} X_i}{n}$ converges completely to 0 if and only if $EX_1 = 0$ and $EX_1^2 < \infty$.*

There has been a vast literature of investigation generalizing and extending this result in many directions. These generalizations and extensions pertain to the partial sums from a sequence (or to the row sums from an array) of either random variables or Banach space valued random elements and are obtained under a variety of different dependence structures. Some of these results indicate the rate of complete convergence in the sense that

$$\sum_{n=1}^{\infty} c_n P(|U_n| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0$$

is proven where $\{c_n, n \geq 1\}$ is a sequence in $(0, \infty)$. Of course, these results only have content when $\sum_{n=1}^{\infty} c_n = \infty$.

We refer the reader to Hu, Rosalsky, and Volodin [3] and Shen, Wang, and Zhu [4] (and references in these articles) for comprehensive reviews of the literature on complete convergence.

In the current work, the main results, Theorems 3.1, 3.2, and 3.4, are complete convergence theorems for weighted row sums from arrays of Banach space valued random elements. In Theorems 3.1 and 3.4, the random elements take values in real separable Rademacher type $p (1 \leq p \leq 2)$ Banach spaces whereas in Theorem 3.2, the random elements take values in real separable martingale type $p (1 \leq p \leq 2)$ Banach spaces. (Technical definitions such as these will be reviewed in Section 2.) Throughout, $\{k_n, n \geq 1\}$ is a sequence of positive integers with $k_n \to \infty$ as $n \to \infty$. In Theorems 3.1 and 3.4, the array is assumed to be comprised of rowwise independent random elements; that is, the random elements from the same row are independent but no independence conditions are imposed between the random elements from different rows. In Theorem 3.2, the array after being suitably centered, is comprised of martingale differences. Theorem 3.4 is a version of Theorem 3.1 with random variable weights.

As will be apparent, the current work owes much to the work of Shen, Wang, and Zhu [4] (especially to their Theorem 7). Shen, Wang, and Zhu [4] corrected, simplified, and extended a previous result of Cai [5]. Despite the brilliance and originality of the Shen, Wang, and Zhu [4] article, we point out that its Theorem 10 is not valid as formulated. One of its hypotheses is

$$\sum_{i=1}^{n} a_i^2 = O\left(\frac{1}{(\log n)^{1+\alpha}}\right) \tag{1.1}$$

for some $\alpha > 0$ where $\{a_n, n \geq 1\}$ is a sequence of constants and its conclusion is of the form

$$\sum_{n=1}^{\infty} \frac{p_n}{n} < \infty \tag{1.2}$$

where $\{p_n, n \geq 1\}$ is a nondecreasing sequence of probabilities. The assertions (1.1) and (1.2) cannot hold if $a_i \neq 0$ for some $i \geq 1$ and $p_n > 0$ for some $n \geq 1$.

2. Preliminaries

Throughout this article, all random elements under consideration are defined on a fixed but otherwise arbitrary probability space $(\Omega, \mathcal{F}, P)$ and take values in a real separable Banach space $\mathcal{X}$ with norm $\|\cdot\|$. It is supposed that $\mathcal{X}$ is equipped with its Borel σ-algebra $\mathcal{B}$; that is, $\mathcal{B}$ is the σ-algebra generated by the class of open subsets of $\mathcal{X}$ determined by $\|\cdot\|$. A *random element* V in $\mathcal{X}$ is an $\mathcal{F}$-measurable transformation from Ω to the measurable space $(\mathcal{X}, \mathcal{B})$. We use the symbol C to denote a generic constant $(0<C<\infty)$ which is not necessary the same one in each appearance. For $x > 0$, we define $\log x$ by $\log x = \log_e x \vee e$ where $\log_e$ is the logarithm to the base e. Technical definitions relevant to the current work will be discussed in this section and the key lemmas which are used to prove the main results will be presented.

The *expected value* or *mean* of a random element V, denoted by EV or by $E(V)$, is defined to be the *Pettis integral* provided it exists; that is, V has expected value $EV \in \mathcal{X}$ if $f(EV) = E(f(V))$ for every $f \in \mathcal{X}^*$, where $\mathcal{X}^*$ is the (*dual*) space of all continuous linear functionals on $\mathcal{X}$. If $E\|V\| < \infty$, then (see, e.g., Taylor [6]) V has an expected value.

Let $\{Y_n, n \geq 1\}$ be a symmetric *Bernoulli sequence*; that is, $\{Y_n, n \geq 1\}$ is a sequence of i.i.d. random variables with $P(Y_1 = 1) = P(Y_1 = -1) = 1/2$. Let $\mathcal{X}^\infty = \mathcal{X} \times \mathcal{X} \times \mathcal{X} \times \cdots$ and define

$$\mathcal{C}(\mathcal{X}) = \left\{ (v_1, v_2, \ldots) \in \mathcal{X}^\infty : \sum_{n=1}^{\infty} Y_n v_n \text{ converges in probability} \right\}.$$

Let $1 \leq p \leq 2$. Then $\mathcal{X}$ is said to be of *Rademacher type p* if there exists a constant $0<C<\infty$ such that

$$E\left\| \sum_{n=1}^{\infty} Y_n v_n \right\|^p \leq C \sum_{n=1}^{\infty} \|v_n\|^p \quad \text{for all } (v_1, v_2, \ldots) \in \mathcal{C}(\mathcal{X}).$$

Rosalsky and Volodin [7] pointed out that the condition that $\mathcal{X}$ is of Rademacher type p is indeed equivalent to the structurally simpler condition that there exists a constant $0<C<\infty$ such that

$$E\left\| \sum_{n=1}^{N} Y_n v_n \right\|^p \leq C \sum_{n=1}^{N} \|v_n\|^p \quad \text{for all } N \geq 1 \quad \text{and } v_n \in \mathcal{X}, 1 \leq n \leq N.$$

Moreover, Hoffmann-Jørgensen and Pisier [8] proved for $1 \leq p \leq 2$ that a real separable Banach space is of Rademacher type p if and only if there exists a constant $0<C<\infty$ such that

$$E\left\| \sum_{i=1}^{n} V_i \right\|^p \leq C \sum_{i=1}^{n} E\|V_i\|^p$$

for every finite collection $\{V_1, \ldots, V_n\}$ of independent mean 0 random elements.

If a real separable Banach space is of Rademacher type p for some $p \in (1, 2]$, then it is of Rademacher type q for all $q \in [1, p]$.

For a random element V and a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$, the *conditional expectation* $E(V|\mathcal{G})$ was introduced by Scalora [9] and is defined analogously to that in the random variable case and enjoys similar properties. See Scalora [9] for a complete development including Banach space valued martingales and martingale convergence theorems.

A real separable Banach space $\mathcal{X}$ is said to be of *martingale type* $p(1 \le p \le 2)$ if there exists a constant $0 < C < \infty$ such that for all martingales $\{S_n, \mathcal{F}_n, n \ge 1\}$ with values in $\mathcal{X}$,

$$\sup_{n\ge1} E\|S_n\|^p \le C\sum_{n=1}^{\infty} E\|S_n - S_{n-1}\|^p$$

where $S_0 \equiv 0$. It can be shown (see Pisier [10] and [11]) that $\mathcal{X}$ being of martingale type p is indeed equivalent to apparently stronger condition that for all $1 \le q < \infty$, there exists a constant $C_{p,q} < \infty$ such that for all martingales $\{S_n, \mathcal{F}_n, n \ge 1\}$ with values in $\mathcal{X}$,

$$E\left(\left(\sup_{n\ge1}\|S_n\|\right)^q\right) \le C_{p,q} E\left(\sum_{n=1}^{\infty}\|S_n - S_{n-1}\|^p\right)^{q/p}. \tag{2.1}$$

It readily follows from (2.1) that if $\mathcal{X}$ is of martingale type p for some $p \in (1, 2]$, then it is of martingale type q for all $q \in [1, p]$.

Every real separable Banach space is of martingale type (at least) 1. For $1 \le p < \infty$, the L_p-spaces and ℓ_p-spaces are of martingale type $p\wedge2$. Detailed discussion concerning martingale type p Banach spaces can be found in Pisier [10] and [11], Woyczyński [12] and [13], and Schwartz [14].

It follows from the Hoffmann-Jørgensen and Pisier [8] characterization of Rademacher type p Banach spaces discussed above that if a Banach space is of martingale type p, then it is of Rademacher type p. But a Banach space can be of Rademacher type 2 (hence be of Rademacher type p for all $p \in [1, 2]$) yet be of martingale type p only for $p = 1$; for details see Pisier [11] and James [15].

The key lemma in the proofs of Theorems 3.1 and 3.4 follows.

Lemma 2.1. (Rosalsky and Van Thanh [16], Lemma 2.1)) *Suppose that the real separable Banach space $\mathcal{X}$ is of Rademacher type $p(1 \le p \le 2)$. Then there exists a constant $C_p \in (0, \infty)$ depending only on p such that for every sequence $\{V_n, n \ge 1\}$ of independent mean 0 random elements,*

$$E\left(\left(\max_{1\le j\le n}\left\|\sum_{i=1}^{j} V_i\right\|\right)^p\right) \le C_p\sum_{i=1}^{n} E\|V_i\|^p, n \ge 1.$$

The key lemma in the proof of Theorem 3.2 follows. The lemma is an immediate consequence of (2.1) taking $q = p$. Alternatively, the lemma follows by the argument used in the proof of Lemma 2.1 of Rosalsky and Van Thanh [16], *mutatis mutandis*.

Lemma 2.2. *Suppose that the real separable Banach space $\mathcal{X}$ is of martingale type $p(1 \le p \le 2)$. Then there exists a constant $C_p \in (0, \infty)$ depending only on p such that for all martingales $\{S_n, \mathcal{F}_n, n \ge 1\}$ with values in $\mathcal{X}$,*

$$E\left(\left(\max_{1\le j\le n}\|S_j\|\right)^p\right) \le C_p\sum_{i=1}^{n} E\|S_i - S_{i-1}\|^p, n \ge 1$$

where $S_0 \equiv 0$.

3. Mainstream

With the preliminaries accounted for, the main results may be stated and proved. In Theorems 3.1 and 3.2, the $a_{n,i}, 1 \le i \le k_n, n \ge 1$ are *weights* and the $S_{n,j}, 1 \le j \le k_n, n \ge 1$ are thus weighted row sums. Theorems 3.1, 3.2, and 3.4 are apparently new results when $\mathcal{X} = \mathbb{R}$.

Theorem 3.1. *Let $\{V_{n,i}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise independent mean 0 random elements in a real separable Rademacher type $p(1 \le p \le 2)$ Banach space and suppose that*

$$\sup\{E\|V_{n,i}\|^p : 1 \le i \le k_n, n \ge 1\} < \infty. \tag{3.1}$$

Let $\{c_n, n \ge 1\}$ be a sequence in $(0, \infty)$ and let $\{a_{n,i}, 1 \le i \le k_n, n \ge 1\}$ be an array of constants such that

$$\sum_{n=1}^{\infty} \frac{c_n}{n} \beta_{n,p} < \infty, \tag{3.2}$$

where $\beta_{n,p} = \sum_{i=1}^{k_n} |a_{n,i}|^p, \ n \ge 1$. Let

$$S_{n,j} = \sum_{i=1}^{j} a_{n,i} V_{n,i}, 1 \le j \le k_n, n \ge 1.$$

Then

$$\sum_{n=1}^{\infty} \frac{c_n}{n} P\left(\max_{1 \le j \le k_n} \|S_{n,j}\| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{3.3}$$

Proof. For arbitrary $\varepsilon > 0$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{c_n}{n} P\left(\max_{1 \le j \le k_n} \|S_{n,j}\| > \varepsilon\right) \\
&\le C \sum_{n=1}^{\infty} \frac{c_n}{n} E\left(\left(\max_{1 \le j \le k_n} \|S_{n,j}\|\right)^p\right) \quad \text{(by the Markov inequality)} \\
&\le C \sum_{n=1}^{\infty} \frac{c_n}{n} \sum_{i=1}^{k_n} E\|a_{n,i} V_{n,i}\|^p \quad \text{(by Lemma 2.1)} \\
&= C \sum_{n=1}^{\infty} \frac{c_n}{n} \sum_{i=1}^{k_n} |a_{n,i}|^p E\|V_{n,i}\|^p \\
&\le C \sum_{n=1}^{\infty} \frac{c_n}{n} \sum_{i=1}^{k_n} |a_{n,i}|^p \quad \text{(by (3.1))} \\
&= C \sum_{n=1}^{\infty} \frac{c_n}{n} \beta_{n,p} < \infty \quad \text{(by (3.2))}
\end{aligned}$$

thereby proving (3.3). □

Remark 3.1. The larger are the $c_n, n \geq 1$, the stronger is the assumption (3.2) as well as the conclusion (3.3).

Corollary 3.1. *If the hypothesis of Theorem 3.1 are satisfied with $n = O(c_n)$, then*

$$\lim_{n\to\infty} \max_{1\leq j\leq k_n} \|S_{n,j}\| = 0 \quad a.s. \tag{3.4}$$

Proof. It follows from $n = O(c_n)$ and (3.3) that

$$\sum_{n=1}^{\infty} P\Big(\max_{1\leq j\leq k_n} \|S_{n,j}\| > \varepsilon\Big) < \infty \qquad \text{for all } \varepsilon > 0.$$

Then by the Borel–Cantelli lemma,

$$P\Big(\max_{1\leq j\leq k_n} \|S_{n,j}\| > \varepsilon \text{ i.o.(n)}\Big) = 0 \qquad \text{for all } \varepsilon > 0$$

thereby proving (3.4). □

Corollary 3.2. *Let $\{V_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent mean 0 random elements in a real separable Rademacher type $p(1 \leq p \leq 2)$ Banach space and suppose that (3.1) holds. Let $\{c_n, n \geq 1\}$ be a sequence in $(0, \infty)$ and let $\{a_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of constants such that*

$$\beta_{n,p} \equiv \sum_{i=1}^{k_n} |a_{n,i}|^p = O\left(\frac{1}{c_n(\log n)(\log\log n)^{1+\delta}}\right) \qquad \textit{for some } \delta > 0. \tag{3.5}$$

Then (3.3) holds.

Proof. Note that (3.2) holds since by (3.5) we have

$$\sum_{n=1}^{\infty} \frac{c_n}{n} \beta_{n,p} \leq C \sum_{n=1}^{\infty} \frac{1}{n(\log n)(\log\log n)^{1+\delta}} < \infty.$$

The conclusion (3.3) follows immediately from Theorem 3.1. □

The following example, which was inspired by an example presented in Kuczmaszewska and Szynal [17], shows that Corollary 3.2 and Theorem 3.1 can fail if the Banach space is not of Rademacher type p where $p \in (1, 2]$.

Example 3.1. Consider the real separable Banach space ℓ_1 of absolutely summable real sequences $v = \{v_k, k \geq 1\}$ with norm $\|v\| = \sum_{k=1}^{\infty} |v_k|$. It is well known that ℓ_1 is not of Rademacher type p for every $p \in (1, 2]$. Let $v^{(i)}$ denote the i-th element of the standard basis in $\ell_1, i \geq 1$; that is, $v^{(i)}$ is the element in ℓ_1 having 1 for its i-th coordinate and 0 for the other coordinates, $i \geq 1$. Let $p \in (1, 2]$ and $\alpha \in \left(\frac{1}{p}, 1\right)$. Define an array $\{V_{n,i}, 1 \leq i \leq n, n \geq 1\}$ of random elements in ℓ_1 by requiring $\{V_{n,i}, 1 \leq i \leq n, n \geq 1\}$ to be a rowwise independent array with

$$P\left(V_{n,i} = v^{(i)}\right) = P\left(V_{n,i} = -v^{(i)}\right) = \frac{1}{2},\ 1 \le i \le n,\ n \ge 1.$$

Let

$$c_n = 1, n \ge 1 \text{ and } a_{n,i} = \frac{1}{n^\alpha}, 1 \le i \le n,\ n \ge 1.$$

Then $E\|V_{n,i}\|^p = 1, 1 \le i \le n, n \ge 1,$ and (3.5) and (3.2) hold with $k_n = n, n \ge 1$. Note that for $n \ge 1$,

$$\|S_{n,n}\| = \left\| \sum_{i=1}^{n} \frac{1}{n^\alpha} V_{n,i} \right\| = \frac{\sum_{i=1}^n 1}{n^\alpha} = n^{1-\alpha} \text{ a.s.} \tag{3.6}$$

Then

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \le j \le k_n} \|S_{n,j}\| > \frac{1}{2}\right) \\ &\ge \sum_{n=1}^{\infty} \frac{1}{n} P\left(\|S_{n,n}\| > \frac{1}{2}\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \quad \left(\text{by } (3.6) \text{ and } 1-\alpha > 0\right) \\ &= \infty \end{aligned}$$

and so the conclusion (3.3) of Corollary 3.2 and Theorem 3.1 fails.

Theorem 3.2. *Let $\{V_{n,i}, 1 \le i \le k_n, n \ge 1\}$ be an array of random elements in a real separable martingale type $p (1 \le p \le 2)$ Banach space $\mathcal{X}$ and suppose that (3.1) holds. Let $\{c_n, n \ge 1\}$ be a sequence in $(0, \infty)$ and let $\{a_{n,i}, 1 \le i \le k_n, n \ge 1\}$ be an array of constants satisfying (3.2). Let*

$$S_{n,j} = \sum_{i=1}^{j} a_{n,i}\left(V_{n,i} - \mu_{n,i}\right), 1 \le j \le k_n, n \ge 1$$

where

$$\mu_{n,1} = EV_{n,1}, \mu_{n,i} = E\left(V_{n,i} | V_{n,1}, \ldots, V_{n,i-1}\right), 2 \le i \le k_n, n \ge 1.$$

Then

$$\sum_{n=1}^{\infty} \frac{c_n}{n} P\left(\max_{1 \le j \le k_n} \|S_{n,j}\| > \varepsilon\right) < \infty \quad \textit{for all } \varepsilon > 0.$$

Proof. Let

$$V_{n,0} = 0 \quad \text{ and } \mathcal{F}_{n,i} = \sigma(V_{n,0}, \ldots, V_{n,i}), 0 \le i \le k_n, n \ge 1.$$

For all $n \ge 1$ and $1 \le i \le k_n$,

$$\mu_{n,i} \text{ is } \mathcal{F}_{n,i-1}\text{-measurable}$$

and hence

$$E\left(a_{n,i}(V_{n,i}-\mu_{n,i})|\mathcal{F}_{n,i-1}\right) = a_{n,i}(\mu_{n,i}-\mu_{n,i}) = 0 \quad \text{a.s.}$$

Thus for all $n \geq 1$, the $a_{n,i}(V_{n,i}-\mu_{n,i}), 1 \leq i \leq k_n$ are martingale differences; that is, $\{S_{n,j}, \mathcal{F}_{n,j}, 1 \leq j \leq k_n\}$ is a martingale in $\mathcal{X}$.

Next, for all $n \geq 1$ and $1 \leq i \leq k_n$,

$$\begin{aligned}
&E\|V_{n,i}-\mu_{n,i}\|^p \\
&= E\|V_{n,i}-E(V_{n,i}|\mathcal{F}_{n,i-1})\|^p \\
&\leq E(\|V_{n,i}\| + \|E(V_{n,i}|\mathcal{F}_{n,i-1})\|)^p \\
&\leq E(\|V_{n,i}\| + E(\|V_{n,i}\||\mathcal{F}_{n,i-1}))^p \quad (\text{by Theorem 2.2 of Scalara [9]}) \\
&\leq CE\left(\|V_{n,i}\|^p + (E(\|V_{n,i}\||\mathcal{F}_{n,i-1}))^p\right) \\
&\leq CE(\|V_{n,i}\|^p + E(\|V_{n,i}\|^p|\mathcal{F}_{n,i-1}))\,(\text{by Jensen's inequality for conditional expectations}) \\
&= CE\|V_{n.i}\|^p.
\end{aligned} \tag{3.7}$$

Then for arbitrary $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{c_n}{n}P\left(\max_{1\leq j\leq k_n}\|S_{n,j}\| > \varepsilon\right) \\
&\leq C\sum_{n=1}^{\infty}\frac{c_n}{n}E\left(\left(\max_{1\leq j\leq k_n}\|S_{n,j}\|\right)^p\right) \qquad (\text{by the Markov inequality}) \\
&\leq C\sum_{n=1}^{\infty}\frac{c_n}{n}\sum_{i=1}^{k_n}E\|a_{n,i}(V_{n,i}-\mu_{n,i})\|^p \quad (\text{by Lemma 2.2}) \\
&\leq C\sum_{n=1}^{\infty}\frac{c_n}{n}\sum_{i=1}^{k_n}|a_{n,i}|^p E\|V_{n,i}\|^p \quad (\text{by (3.7)})
\end{aligned}$$

and the rest of the argument proceeds exactly as in the proof of Theorem 3.1. □

Corollary 3.3. *If the hypothesis of Theorem 3.2 are satisfied with $n = O(c_n)$, then*

$$\lim_{n\to\infty}\max_{1\leq j\leq k_n}\|S_{n,j}\| = 0 \quad a.s.$$

Proof. The argument is identical to that for proving Corollary 3.1. □

Corollary 3.4. *Let $\{V_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of random elements in a real separable martingale type $p(1 \leq p \leq 2)$ Banach space $\mathcal{X}$ and suppose that (3.1) holds. Let $\{c_n, n \geq 1\}$ be a sequence in $(0, \infty)$ and let $\{a_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of constants satisfying (3.5). Let*

$$S_{n,j} = \sum_{i=1}^{j} a_{n,i}(V_{n,i}-\mu_{n,i}), 1 \leq j \leq k_n, n \geq 1$$

where

$$\mu_{n,1} = EV_{n,1}, \mu_{n,i} = E\big(V_{n,i}|V_{n,1}, \ldots, V_{n,i-1}\big), 2 \le i \le k_n, n \ge 1.$$

Then

$$\sum_{n=1}^{\infty} \frac{c_n}{n} P\Big(\max_{1\le j\le k_n} \|S_{n,j}\| > \varepsilon\Big) < \infty \quad \text{for all } \varepsilon > 0.$$

Proof. The argument is identical to that for proving Corollary 3.2 except that Theorem 3.2 is employed instead of Theorem 3.1. □

The following complete convergence theorem is the main result in the article by Hu, Rosalsky, and Volodin [3]. In Example 3.2 below, the hypotheses of Corollary 3.2 are satisfied but those of Theorem 3.3 are not whereas in Example 3.3 below, the hypotheses of Theorem 3.3 are satisfied but those of Theorem 3.1 are not.

Theorem 3.3. *(Hu, Rosalsky, and Volodin [3], Theorem 3.1). Let $\{W_{n,i}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise independent random elements taking values in a real separable Rademacher type $p (1 \le p \le 2)$ Banach space and let $\{c_n, n \ge 1\}$ be a sequence of positive constants. Suppose for some $J > 0$ and some $\delta_1, \delta_2 > 0$ that*

$$\sum_{n=1}^{\infty} \frac{c_n}{n} \sum_{i=1}^{k_n} P\big(\|W_{n,i}\| > \varepsilon\big) < \infty \quad \text{for all } \varepsilon > 0, \tag{3.8}$$

$$\sum_{n=1}^{\infty} \frac{c_n}{n} \left(\sum_{i=1}^{k_n} E\Big(\|W_{n,i}\|^p I\big(\|W_{n,i}\| \le \delta_1\big)\Big)\right)^J < \infty, \tag{3.9}$$

and

$$\sum_{i=1}^{k_n} E\big(W_{n,i} I\big(\|W_{n,i}\| \le \delta_2\big)\big) \to 0. \tag{3.10}$$

Then

$$\sum_{n=1}^{\infty} \frac{c_n}{n} P\left(\left\|\sum_{i=1}^{k_n} W_{n,i}\right\| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0.$$

Example 3.2. Let $\mathcal{X}$ be a real separable Rademacher type $p = 2$ Banach space, let

$$k_n = n, c_n = 1, a_n = \frac{1}{\sqrt{n \log n} \log \log n}, \quad n \ge 1,$$

and let

$$a_{n,i} = a_n, \quad 1 \le i \le n, n \ge 1.$$

Let $\{V_{n,i}, 1 \le i \le n, n \ge 1\}$ be an array of identically distributed and rowwise independent random elements with $E\|V_{1,1}\|^2 < \infty$ and $EV_{1,1} \ne 0$. Then $E\|V_{n,i}\|^2 = E\|V_{1,1}\|^2, 1 \le i \le n, n \ge 1$. Moreover,

$$\sum_{i=1}^{n}|a_{n,i}|^2 = \frac{n}{(n\log n)(\log\log n)^2} = O\left(\frac{1}{c_n(\log n)(\log\log n)^2}\right).$$

The hypotheses of Corollary 3.2 are satisfied with $\delta = 1$ and so by Corollary 3.2,

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\frac{\max_{1\le j\le n}\|\sum_{i=1}^{j}V_{n,i}\|}{\sqrt{n\log n}\log\log n} > \varepsilon\right)$$
$$= \sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le j\le n}\left\|\sum_{i=1}^{j}a_{n,i}V_{n,i}\right\| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0.$$

Next, let $W_{n,i} = a_{n,i}V_{n,i}, 1 \le i \le n, n \ge 1$. We will verify that condition (3.10) of Theorem 3.3 fails for all $\delta_2 > 0$. For $\delta_2 > 0$, it follows from $E\|V_{1,1}\| < \infty, 0 < a_n \to 0$, and the Lebesgue dominated convergence theorem that

$$\left\|E\left(V_{1,1}I\left(\|V_{1,1}\| < \frac{\delta_2}{a_n}\right)\right) - EV_{1,1}\right\|$$
$$= \left\|E\left(V_{1,1}I\left(\|V_{1,1}\| \ge \frac{\delta_2}{a_n}\right)\right)\right\|$$
$$\le E\left(\|V_{1,1}\|I\left(\|V_{1,1}\| \ge \frac{\delta_2}{a_n}\right)\right) \to 0.$$

Thus

$$E\left(V_{1,1}I\left(\|V_{1,1}\| < \frac{\delta_2}{a_n}\right)\right) \to EV_{1,1} \ne 0$$

and so

$$\left\|E\left(V_{1,1}I\left(\|V_{1,1}\| < \frac{\delta_2}{a_n}\right)\right)\right\| \to \|EV_{1,1}\| > 0.$$

Then

$$\left\|\sum_{i=1}^{k_n}E(W_{n,i}I(\|W_{n,i}\| \le \delta_2))\right\|$$
$$= \left\|\sum_{i=1}^{k_n}a_nE\left(V_{n,i}I\left(\|V_{n,i}\| < \frac{\delta_2}{a_n}\right)\right)\right\|$$
$$= \left\|na_nE\left(V_{1,1}I\left(\|V_{1,1}\| < \frac{\delta_2}{a_n}\right)\right)\right\|$$
$$= \frac{\sqrt{n}}{\sqrt{\log n}\log\log n}\left\|E\left(V_{1,1}I\left(\|V_{1,1}\| < \frac{\delta_2}{a_n}\right)\right)\right\| \to \infty.$$

Thus (3.10) fails for all $\delta_2 > 0$.

Example 3.3. Let $\{V_{n,i}, 1 \le i \le n, n \ge 1\}$ be an array of identically distributed and rowwise independent symmetric random variables with $E|V_{1,1}| < \infty$. Let $p = 1, k_n = n, c_n = 1, a_{n,i} =$

$\frac{1}{n\log n}$, $W_{n,i} = a_{n,i}V_{n,i}, 1 \le i \le n, n \ge 1$, and let $\delta_1 = \delta_2 = 1$ and $J > 1$. We first verify that conditions (3.8), (3.9), and (3.10) are satisfied.

For arbitrary $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{c_n}{n} \sum_{i=1}^{n} P\left(|W_{n,i}| > \varepsilon\right) = \sum_{n=1}^{\infty} \frac{1}{n} nP\left(|V_{1,1}| > (n\log n)\varepsilon\right) \le \sum_{n=1}^{\infty} P\left(|V_{1,1}| > n\varepsilon\right) < \infty$$

since $E|V_{1,1}| < \infty$ thereby verifying (3.8).

Next,

$$\sum_{n=1}^{\infty} \frac{c_n}{n} \left(\sum_{i=1}^{n} E\left(|W_{n,i}|^p I\left(|W_{n,i}| \le \delta_1\right)\right)\right)^J \le \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{i=1}^{n} \frac{E|V_{1,1}|}{n\log n}\right)^J = \sum_{n=1}^{\infty} \frac{\left(E|V_{1,1}|\right)^J}{n(\log n)^J} < \infty$$

since $J > 1$ thereby verifying (3.9).

Finally, the condition (3.10) holds by the symmetry hypothesis.

Thus by Theorem 3.3,

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\left|\sum_{i=1}^{n} V_{n,i}\right|}{n\log n} > \varepsilon \right) = \sum_{n=1}^{\infty} \frac{1}{n} P\left(\left| \sum_{i=1}^{n} W_{n,i} \right| > \varepsilon \right) < \infty \quad \text{for all } \varepsilon > 0.$$

But

$$\sum_{i=1}^{n} |a_{n,i}|^p = \frac{1}{\log n}$$

and so the condition (3.2) of Theorem 3.1 fails.

Remark 3.2. In Example 3.2, the hypotheses of Theorem 3.1 are also satisfied and in Example 3.3, the hypotheses of Corollary 3.2 are not satisfied.

Remark 3.3. It is an open question as to whether or not (3.3) holds for the random variables in Example 3.3. However, in the following example, the condition (3.2) of Theorem 3.1 is also not satisfied yet its conclusion (3.3) does hold. The example is a modification of Example 3.1.

Example 3.4. Let $\{V_{n,i}, 1 \le i \le n, n \ge 1\}$ be the array of random elements taking values in the real separable Rademacher type $p = 1$ Banach space ℓ_1 considered in Example 3.1. Let $\{c_n, n \ge 1\}$ be a sequence of constants such that $\inf_{n\ge1} c_n > 0$ and let $k_n = n, a_{n,i} = \frac{1}{n\log n}, 1 \le i \le n, n \ge 1$. As in Example 3.3, the condition (3.2) of Theorem 3.1 fails. Let $S_{n,j}, 1 \le j \le n, n \ge 1$ be as in Theorem 3.1. Note that for all $n \ge 1$,

$$\max_{1\le j\le n} \|S_{n,j}\| = \frac{\max_{1\le j\le n} \left\| \sum_{i=1}^{j} V_{n,i} \right\|}{n\log n} = \frac{\max_{1\le j\le n} \sum_{i=1}^{j} 1}{n\log n} = \frac{1}{\log n} \quad \text{a.s.}$$

and so for all $\varepsilon > 0$ and all large n

$$P\left(\max_{1\le j\le n} \|S_{n,j}\| > \varepsilon \right) = 0.$$

Consequently, (3.3) holds.

We close by presenting a version of Theorem 3.1 where the weights $A_{n,i}, 1 \leq i \leq k_n, n \geq 1$ are random variables. In view of the randomness often encountered in the applied sciences, it has become increasingly important to establish limit theorems for randomly weighted sums. For example, Rosalsky and Sreehari [18] provided an application of randomly weighted sums to the field of queueing theory. Taylor and Padgett [19], Wei and Taylor [20] and [21], Taylor and Calhoun [22]), Taylor, Raina, and Daffer [23], Ordóñez Cabrera [24], Adler, Rosalsky, and Taylor [25], and Rosalsky, Sreehari, and Volodin [26] studied the limiting behavior of randomly weighted sums in real separable Banach spaces.

In Theorem 3.4, there are no independence or uncorrelation conditions between the random weight $A_{n,i}$ and the random element $V_{n,i}, 1 \leq i \leq k_n, n \geq 1$. It is assumed that $\{A_{n,i}V_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ is an array of rowwise independent mean 0 random elements. It should be noted that each $A_{n,i}V_{n,i}$ is automatically a random element (see, e.g., Taylor [6], p. 24).

Theorem 3.4. *Let $\{V_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements taking values in a real separable Rademacher type $p(1 \leq p \leq 2)$ Banach space. Let $\{c_n, n \geq 1\}$ be a sequence of constants and let $\{A_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of random variables. Suppose that the array $\{A_{n,i}V_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ is comprised of rowwise independent mean 0 random elements. Let*

$$S_{n,j} = \sum_{i=1}^{j} A_{n,i}V_{n,i}, 1 \leq j \leq k_n, n \geq 1.$$

Suppose for some $q > 1$ that

(i)

$$\sup\left\{E\|V_{n,i}\|^{\frac{pq}{q-1}} : 1 \leq i \leq k_n, n \geq 1\right\} < \infty \tag{3.11}$$

and

$$\sum_{n=1}^{\infty} \frac{c_n}{n} \sum_{i=1}^{k_n} \left(E|A_{n,i}|^{pq}\right)^{1/q} < \infty \tag{3.12}$$

or

(ii)

$$\sup\{E\|V_{n,i}\|^{pq} : 1 \leq i \leq k_n, n \geq 1\} < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{c_n}{n} \sum_{i=1}^{k_n} \left(E|A_{n,i}|^{\frac{pq}{q-1}}\right)^{\frac{q-1}{q}} < \infty$$

or

(iii)

$$\sup\left\{E\|A_{n,i}\|^{\frac{pq}{q-1}} : 1 \leq i \leq k_n, n \geq 1\right\} < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{c_n}{n} \sum_{i=1}^{k_n} \left(E\|V_{n,i}\|^{pq}\right)^{1/q} < \infty$$

or

(iv)

$$\sup\left\{E\|A_{n,i}\|^{pq} : 1 \le i \le k_n, n \ge 1\right\} < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{c_n}{n} \sum_{i=1}^{k_n} \left(E\|V_{n,i}\|^{\frac{pq}{q-1}}\right)^{\frac{q-1}{q}} < \infty$$

holds. Then

$$\sum_{n=1}^{\infty} \frac{c_n}{n} P\left(\max_{1\le j\le k_n} \|S_{n,j}\| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{3.13}$$

Proof. The proof will only be given for (i) since the arguments for (ii), (iii), and (iv) are similar. For arbitrary $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{c_n}{n} P\left(\max_{1\le j\le k_n} \|S_{n,j}\| > \varepsilon\right) \\
&\le C\sum_{n=1}^{\infty} \frac{c_n}{n} E\left(\left(\max_{1\le j\le k_n} \|S_{n,j}\|\right)^p\right) \quad (\text{by the Markov inequality}) \\
&\le C\sum_{n=1}^{\infty} \frac{c_n}{n} \sum_{i=1}^{k_n} E\|A_{n,i}V_{n,i}\|^p \quad (\text{by Lemma 2.1}) \\
&\le C\sum_{n=1}^{\infty} \frac{c_n}{n} \sum_{i=1}^{k_n} \left(E|A_{n,i}|^{pq}\right)^{\frac{1}{q}} \left(E\|V_{n,i}\|^{\frac{pq}{q-1}}\right)^{\frac{q-1}{q}} \quad (\text{by Hölder's inequality}) \\
&\le C\sum_{n=1}^{\infty} \frac{c_n}{n} \sum_{i=1}^{k_n} \left(E|A_{n,i}|^{pq}\right)^{1/q} \quad (\text{by (3.11)}) \\
&< \infty \quad (\text{by (3.12)})
\end{aligned}$$

thereby proving (3.13). □

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